Math 347 Worksheet

Lecture 18: Monotone convergence and properties of limits

October 15, 2018

1. Let $x_n = \frac{1+n}{1+2n}$.
   (i) Prove that the limit $\lim x_n$ exists by using the monotone convergence theorem.
   (ii) Prove that $\lim x_n = 2$ using the definition of limit.

2. If a_n and b_n are two nondecreasing sequences, and
$$\lim(a_n - b_n) = 0.$$
Then both sequences converge and have the same limit. Prove that this is not the case if they are not monotone, i.e. nondecreasing.

3. (The Squeeze Theorem) suppose that a_n, b_n and c_n are three sequences, such that
$$a_n \leq b_n \leq c_n$$
for all n. Suppose that $\lim a_n = L$ and $\lim c_n = L$. Prove that $\lim b_n = L$.

4. Suppose that $x_n \to 0$ and that $|y_n| \leq 1$ for $n \in \mathbb{N}$.
   (i) Find the flaw in the following computation for $\lim x_n y_n$.
$$\lim x_n y_n = \lim x_n \cdot \lim y_n = 0 \cdot \lim y_n = 0.$$
   (ii) Find a valid proof of the fact that $\lim x_n y_n = 0$.

5. (Sequence of rational numbers converging to $\sqrt{2}$). Consider x_1 any rational number such that $x_1^2 > 2$. For any $n \geq 2$, define
$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right).$$
   (i) Prove that if the sequence (x_n) converges, its limit is $\sqrt{2}$.
   (ii) Use the monotone convergence theorem to argue that the sequence (x_n) converges.
   (iii) Let $S = \{x \in \mathbb{Q} \mid x^2 > 2\}$, prove that $\inf(S) = \sqrt{2}$.